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Exactly solvable models of conformal-invariant quantum field theory in *D*-dimensional space

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Dedicated to I.M. Gelfand on his 75th birthday

Abstract. *The method for exact solution of a certain class of models of conformal quantum field theory in D-dimensional Euclidean space is proposed. The method allows one to derive closed differential equations for all the Green functions and also algebraic equations to scale dimensions of all field. A scalar field P of a scale dimension* $d_P = D - 2$ *is needed for nontrivial solutions to exist. At* $D \neq 2$ *this field is converted to a constant that coincides with the central charge of two-dimensional theories. A new class of* $D = 2$ *models has been obtained, where the infinite-parametric symmetry is not manifest. The two-dimensional Wess-Zumino model is used to illustrate the method of solution.*

We take it as a great honour to have this work published in the collection commemorated to the 75-th birthday of one of the most prominent mathematicians of the present Israel Moiseevich Gel'fand, whose fundamental works are widely used in quantum theory of fields and statistical physics.

We would like to express our admiration to his classical results and thank him cordially for his attention goodwill and kind-heartedness.

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I. INTRODUCTION

There is an ample class of models of the quantum field theory in D-dimensional space for which an exact conformal-invariant solution can be built. These models are equivalent to Lagrangian models from a certain viewpoint described below. A method for their exact solution is proposed in the present paper. A closed set of differential equations for all the Green functions, as well as algebraic equations for the scale dimensions of the fields, will be obtained in each model.

The method is based on several statements proved in [1], see also [2, 3]. The Thirring model is simplest in this class of models. Its solution using the described method is presented in [1, 3]. Here we mainly discuss the theories in the D-dimensional space. A new class of models that have not been considered before, is derived at $D = 2$. Some of these models coincide with the minimal models of refs. [4, 5]. This is true particularly, for the two-dimensional Ising and Wess-Zumino models. The infinite-parameter symmetry in the rest of the $D = 2$ models is not realized.

The models in D-dimensional space with $D > 2$ have a peculial feature: non-trivial solutions are possible only if there is a scalar field P of scale dimension $d_P = D - 2$. This field is analogous to the central charge (*) of the $D = 2$ theories. It appears in the commutator of the energy-momentum tensor components. Normalization of the 3-point Green function $\langle P\varphi\varphi\rangle$, where φ is a fundamental field, is one of the theory parameters which are calculated in the course of solution. The field $P(x)$ becomes a constant at $D = 2$, and this parameter coincides with the central charge.

II. PRINCIPAL STATEMENTS

1. In any conformal quantum theory of the scalar field $\varphi(x)$, irrespective of the type of interaction, there exists an infinite set of symmetrical traceless tensor fields [1, 2]

$$P_s \equiv P^{d+x}_{\mu_1 \dots \mu_s}(x)$$

induced by the current of the energy-momentum tensor. These fields appear in the operator expansions of the products or

$$T_{\mu\nu}(x_1)\varphi(x_2) \quad \text{or} \quad j_\mu(x_1)\varphi(x_2)$$

in powers of the difference x_{12}. Scale dimensions of the fields P_s are equal to

(*) The central charge has been discovered by Gel'fand and Fuks (see Functz. Analiz. 2 (1968) 92).

$$d_s = d + s.$$

At $s = 0$ the field P_s coincides with the fundamental one: $P_s|_{s=0} = \varphi(x)$; the vector field P_μ^{d+1} is involved in expansion of the product $j_\mu \varphi$ only, but not of $T_{\mu\nu}\,\varphi$. The proof that these fields exist is based [2] on the Ward identity.

2. It can be shown [1, 6, 7] that every operator equality:

$$(2.1) \qquad P_s = 0 \qquad \text{for a certain } s$$

where P_s is the field from the expansion of the product $T_{\mu\nu}(x_1)\,\varphi(x_2)$ is equivalent to a set of closed linear differential equations for all the Green functions of fundamental and composite fields. More exactly: each equation

$$(2.2) \qquad \langle P_s(x_1)\,\varphi(x_2)\ldots\varphi(x_n)\rangle = 0$$

is equivalent to several differential equations of the form

$$(2.3) \qquad L^{(s)}\left(x, \frac{\partial}{\partial x}\right)\langle\varphi(x_1), \ldots, \varphi(x_n)\rangle = 0$$

where $L^{(s)}(x, \partial/\partial x)$ is a differential operator of the $(s + 1)$ order depending on all the coordinates. Its form can be calculated [1, 2, 7] from the Ward identities. It is very complicated in the general case. The results are given below for the simplest $D = 2$ models.

3. Equations (2.2) are true provided that composite fields O_α are present as well. In case of 3-point functions we have [1, 3]:

$$\langle P_s\,\varphi O_\alpha\rangle = L_\alpha^{(s)}\,\langle\varphi\varphi\,O_\alpha\rangle = 0$$

where $L_\alpha^{(s)}$ is a differential operator. Since the coordinate dependence of the Green functions $\langle\varphi\varphi\,O_\alpha\rangle$ is known we determine algebraic relations for the dimensions d and d_α of the fields φ and O_α.

The scalar field P^{D-2} with the scale dimension $d_P = D - 2$ is especially important among the composite fields. It appears as an operator-valued (at $D > 2$) Schwinger term in the commutator of the energy-momentum tensor components and contributes to the Ward identity for the Green function $\langle\varphi T_{\mu\nu}\,\varphi\,T_{\rho\sigma}\rangle$, see (9.1). It is essential that this is the only Schwinger term compatible with the conformal invariance. We have:

$$(2.4) \qquad \langle\varphi(x_1)\,\varphi(x_2)P^{D-2}\,(x_3)\rangle =$$

$$= - \frac{1}{144\,\pi}\, C \cdot \left(\frac{x_{12}^2}{x_{13}^2 x_{23}^2} \right)^{\frac{D-2}{2}} \langle \varphi(x_1)\, \varphi(x_2) \rangle \tag{2.4}$$

where C is a constant that can be found from the equation

$$\langle P_s\, \varphi T_{\mu\nu} \rangle = L_T^{(s)} \left(x, \frac{\partial}{\partial x} \right) \langle \varphi\, \varphi T_{\mu\nu} \rangle = 0 \tag{2.5}$$

and the Ward identity for the Green function $\langle \varphi T_{\mu\nu}\, \varphi T_{\rho\sigma} \rangle$, see Sec. IX.

For $D = 2$ the field P^{D-2} becomes a constant: $P^{D-2}|_{D=2} = -1/144\,\pi\, C$, where C coincides with the central charge of two-dimensional theories (its values for models of the given class are presented in Section X).

There is another composite scalar field $\chi \sim \varphi^2$, with anomalous dimension Δ. Similarly to (2.5) we have:

$$\langle P_s \varphi P^{D-2} \rangle = L_P^{(s)} \langle \varphi\varphi P^{D-2} \rangle = 0;$$

$$\langle P_s \varphi\chi \rangle = L_\chi^{(s)} \langle \varphi\varphi\chi \rangle = 0$$

The first of these equations is fulfilled if $D > 2$, since $P^{D-2}|_{D=2} = \text{const}$. It can be shown that using (2.5), two relations for the scale dimensions d and Δ can be obtained from the equations. One of them is (for $D > 2$):

$$\frac{f_1^{(s)}(d, \Delta)}{f_1^{(s)}(d, D-2)} = \frac{f_2^{(s)}(d, \Delta)}{f_2^{(s)}(d, D-2)} \tag{2.6}$$

where

$$f_1^{(s)}(d, \Delta) = \frac{\Gamma\left(\frac{\Delta + 2s}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)} \left[\frac{d}{D}(2d + s - D) + \frac{1}{2}(\Delta - 2d) - \right.$$

$$\left. - \frac{1}{4}\, \frac{\Delta(\Delta - 2)(D - \Delta)}{(D + 2s)} - \frac{1}{4}\, \frac{\Delta(\Delta + 2s)(\Delta - D + 2)}{(D + 2s)} \right] +$$

$$+ (-1)^s \frac{\Gamma\left(\frac{2d - \Delta + 2s}{2}\right)}{\Gamma\left(\frac{2d - \Delta}{2}\right)} \left[\frac{\Delta}{D}(2d + s - D) - \frac{\Delta}{2} - \right.$$

$$-\frac{1}{4}\frac{\Delta(2d-\Delta-2)(D-2d+\Delta)}{(D+2s)}-\frac{1}{4}$$

$$\left.\frac{\Delta(2d-\Delta+2s)(2d-\Delta-D+2)}{(D+2s)}\right];$$

$$\frac{1}{s}f_2^{(s)}(d,\Delta)=\frac{\Gamma\left(\frac{\Delta+2s}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)}\left[\frac{d}{D}(2d-D+2)+(\Delta-2d)-\right.$$

$$\left.-\frac{1}{2}\Delta(\Delta-D+2)\right]+(-1)^s\frac{\Gamma\left(\frac{2d-\Delta+2s}{2}\right)}{\Gamma\left(\frac{2d-\Delta}{2}\right)}$$

$$\left[\frac{\Delta}{D}(2d-D+2)-\Delta-\frac{1}{2}\Delta(2d-\Delta-D+2)\right].$$

4. Every operator equality (2.1) can be derived from a certain Lagrangian theory regularized in an appropriate way. [6, 7].

The Thirring model is the simplest case for illustration of the above statement. The spinor field ψ_1 with the dimension $d_1 = d + 1$ that appears in the operator expansion of the product $\psi(x) j_\mu(x+\epsilon)$ makes an analog of the field P_s in the model. The operator equation $\psi_1 = 0$ completely determined [1, 3] the solution of the model and is a direct consequence of the field equation $\hat{\partial}\psi = \lambda\gamma_\mu j_\mu \psi$. A similar situation is also observed in the Wess-Zumino model, see Section XI.

III. FIELD EQUATIONS

The theories where the fields P_s are generated by the operator product $\varphi(x_1)\ T_{\mu\nu}(x_2)$ refer to another class. Let us consider, for example, the theory with the interaction $L_{int} = \lambda\varphi^4$ in the D-dimensional space, where φ is a neutral field. The theory, in this case, requires being further defined. Let us regularize the total Lagrangian by adding the term $\Lambda/2\ (\Box\varphi)^2$, where Λ is the regularization parameter, $\Lambda \to 0$. The solution of this theory is determined according to [6, 7] by equation (2.1) with $s = 2$. Really, let us write the field equation in the form:

(3.1) $$z_2(\Lambda\,\Box^2\varphi + \Box\varphi - \delta m^2\varphi) = \lambda z_1 \varphi\chi$$

where $\chi \sim \varphi^2$ is an intermediate field, and the renormalization constants $z_{1,2}$ and δm^2 depend on the parameter Λ. The operator product $\varphi(x)\,\chi(x)$ in the right-hand side is finite as long as $\Lambda \neq 0$, and diverges when $\Lambda \to 0$. The constant z_1 tends to zero together with Λ; thus we have an ambiguity $0 \times \infty$ in the right-hand side. To resolve it the product $\varphi(x)\,\chi(x)$ should be found when $\Lambda \neq 0$. This product can be defined by setting the arguments of the fields apart so that they be separated by the Λ-depending vector $\epsilon_\mu = \epsilon_\mu(\Lambda)$:

$$\varphi(x)\,\chi(x)\big|_{reg} = \int d\Omega_\epsilon\, \varphi(x)\,\chi(x+\epsilon) \tag{3.2}$$

where

$$\int d\Omega_\epsilon$$

implies averaging over the angles of the vector ϵ_μ. It follow from considerations of dimension that

$$\epsilon^2 \sim \Lambda \tag{3.3}$$

and for the renormalization constants we have

$$z_1 \sim \Lambda^{d - \frac{\Delta}{2} - \frac{D}{2}},\; z_2 \sim \Lambda^{d - \frac{D}{2} + 1},\; z_3 \sim \Lambda^{\Delta - \frac{D}{2}},\; \delta m^2 \sim \frac{1}{\Lambda} \tag{3.4}$$

In the theory thus defined the role of the regularized field equations reduces to fixing the first terms in the operator expansion of the product of the fundamental fields $\varphi(x)\,\chi(x+\epsilon)$. Really, after substituting (3.2) - (3.4) into (3.1) the field equation can be written as

$$\int d\Omega_\epsilon\, \varphi(x)\,\chi(x+\epsilon) =$$

$$= (\epsilon^2)^{-\frac{\Delta}{2}} \{a_0 \varphi(x) + a_1 \epsilon^2 \Box\, \varphi(x) + a_2 (\epsilon^2)^2 \Box^2\, \varphi(x)\}$$

On the other hand, using the contribution of the field $P^{d+2}_{\mu\nu}$ into the above expansion in which the Ward identity is responsible we would get an additional term $\partial_\mu\, \partial_\nu\, P^{d+2}_{\mu\nu}$ in the right-hand side

$$(\epsilon^2)^{-\frac{\Delta}{2}} \{\ldots + a_2 (\epsilon^2)^2\, [\Box^2\, \varphi(x) + \partial_\mu \partial_\nu P^{d+2}_{\mu\nu}(x)\}$$

It must not be there according to the equation. Therefore, it should be required that

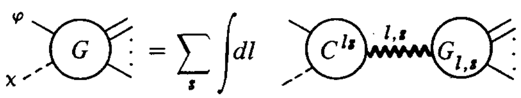
$$P^{d+2}_{\mu\nu}(x) = 0$$

A certain class of more complicated Lagrangians, each of them with several scalar fields $\varphi(x)$ can be treated in a similar way. For each of them the theory solution is defined by one of the equation (2.1) with $s > 2$.

IV. CONFORMAL EXPANSION OF THE GREEN FUNCTIONS OF THE ENERGY-MOMENTUM TENSOR

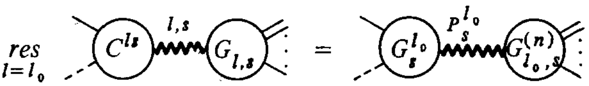
1. Each of the Green functions can be presented as a conformal partial wave expansion. For scalar fields we have:

(4.1)



where $C^{ls} \sim \langle\varphi\chi P^l_s\rangle$ are invariant functions, P^l_s is a tensor field of dimension l and spin s. Each pole of the Kernel $G_{l,s}$ in the point $l = l_0$ in (4.1) corresponds to the tensor field $P^{l_0}_s$ in the operator expansion of the product $\varphi(x)\,\chi(x+\epsilon)$

(4.2)



where

$$G^{l_0}_s = \langle\varphi\chi P^{l_0}_s\rangle,\quad G^{(n)}_{l_0,s} = \operatorname*{res}_{l=l_0} G_{l,s} = \langle P^{l_0}_s\,\varphi\ldots\varphi\rangle$$

This result is the bases of all previous applications of the method of conformal partial wave expansions, see e.g. [1, 9] and the references therein. In particular, separating only the contribution of the field $P^{l_0}_s$ into the operator expansion $\varphi(x)\,\chi(x+\epsilon)$ at $\epsilon \to 0$ we have:

$$\langle\varphi(x)\,\chi(x+\epsilon)\,\varphi\ldots\varphi\rangle\big|_{\epsilon\to 0} \to \int dy\ \tilde{Q}^{l_0 s}(x, x+\epsilon, y)\,\langle P^{l_0}_s(y)\varphi\ldots\varphi\rangle \tag{4.4}$$

where

$$C^{l,s} = \frac{1}{\pi \sin \pi \left(l - \frac{D}{2} \right)} \left\{ Q^{l,s}(x_1 x_2 x_3) - \int \Delta^{l,s}(x_3 y)\, \tilde{Q}^{l,s}(x_1 x_2 y) dy \right\}$$

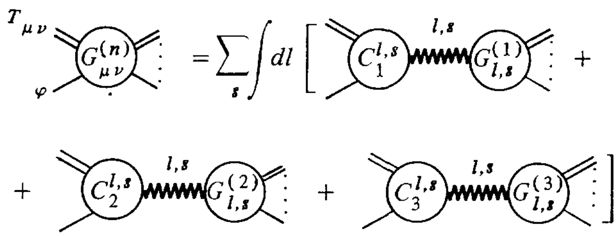
$\Delta^{l,s}$ being the invariant propagator.

2. There is a more complicated situation where the Green function (4.1) contains some tensor field instead of χ. The conformal invariance admits several different functions $C^{l,s}$ when the external field is a tensor. For example, for the tensor of rank two there exist three types of functions $C^{l,s}$ (at $s \geqslant 2$):

(4.5) $$\langle P^l_s \varphi T_{\mu\nu} \rangle = \alpha C^{l,s}_{1\mu\nu} + \beta C^{l,s}_{2\mu\nu} + \gamma C^{l,s}_{3\mu\nu}$$

where α, β, γ are arbitrary constants. Explicit expressions for these functions are given in [2, 8]. The analog to the expansion (4.1) in this case is:

(4.6)



If all of the three kernels $G^{(i)}_{ls}$ have a common pole in the same point l_0, then there are three different fields $P^{(i)}_{l_0 s}$ of the same dimension l_0. The Green functions of these fields (for a given s and different i's)

(4.7) $$\langle P^{(i)}_{l_0 s} \varphi \ldots \varphi \rangle, \ \langle P^{(i)}_{l_0 s} \varphi T_{\mu\nu} \rangle, \quad i = 1, 2, 3$$

are different. However, the functions

(4.8) $$\langle P^{(1)}_{l_0 s} \varphi\chi \rangle \sim \langle P^{(2)}_{l_0 s} \varphi\chi \rangle \sim \langle P^{(3)}_{l_0 s} \varphi\chi \rangle$$

have the same coordinate dependence. The same thing is true for the two-point Green functions

(4.9) $$\langle P^{(1)}_{l_0 s} P^{(1)}_{l_0 s} \rangle = \langle P^{(2)}_{l_0 s} P^{(2)}_{l_0 s} \rangle = \langle P^{(3)}_{l_0 s} P^{(3)}_{l_0 s} \rangle.$$

We assume besides that

(4.9a) $$\langle P^{(i)}_{l_0 s} P^{(k)}_{l_0 s} \rangle = 0 \quad \text{for} \quad i \neq k$$

3. Let us consider the contributions of the fields P_s into (4.6). We are going to show that the Green functions $\langle P^{(i)}_s \varphi T_{\mu\nu} \rangle$ can be presented as a certain

combination of a power function with two types of quasilocal terms (see (4.14)). Indeed, the conformal-invariant Green functions $C_i^{l,s} \equiv C^{ls}_{i\mu\nu}$ have the following structure. All of them have poles (see [2, 10]) in $l = d + s$ due to the factor

$$(x_{13}^2)^{-\frac{D+l-d-s}{2}}$$

We have [11]

$$(x^2)^{-\frac{D+\epsilon}{2}}\Big|_{\epsilon\to 0} \simeq -\frac{1}{\epsilon}\frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\,\delta(x) \tag{4.10}$$

It can be shown [8, 10], that the residues in the poles meet the relation

$$D(D-2)\underset{l=d+s}{res}\, C^{ls}_{1\mu\nu} + D\underset{l=d+s}{res}\, C^{ls}_{2\mu\nu} + \underset{l=d+s}{res}\, C^{ls}_{3\mu\nu} = 0 \tag{4.11}$$

Therefore, there is a regular combination

$$C^{reg}_{\mu\nu} = D(D-2)\,C^{ls}_{1\mu\nu} + DC^{ls}_{2\mu\nu} + C^{ls}_{3\mu\nu} \tag{4.12}$$

equal to [10]

$$C^{reg}_{\mu\nu}\,\big|_{l=d+s} = \langle P_s(x_1)\varphi(x_2)\,T_{\mu\nu}(x_3)\rangle = \{\ldots\} + \tag{4.12a}$$

$$(x^2)^{\frac{D}{2}-d}(x_{23}^2)^{\frac{D-2}{2}}\left[\overleftrightarrow{\partial}^{x_3}_{\mu}\,\overleftrightarrow{\partial}^{x_3}_{\gamma} - \frac{2}{(D-2)}\left(\overleftarrow{\partial}^{x_3}_{\mu}\,\overrightarrow{\partial}^{x_3}_{\nu} + \overleftarrow{\partial}^{x_3}_{\nu}\,\overrightarrow{\partial}^{x_3}_{\mu}\right) -\right.$$

$$\left.-\ trace\right](x_{13}^2)^{-\frac{D-2}{2}}\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2),$$

$$\overleftrightarrow{\partial}_{\mu} = \overrightarrow{\partial}_{\mu} - \overleftarrow{\partial}_{\mu},$$

$$\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3\,x_2) = \lambda^{x_1}_{\mu_1}(x_3\,x_2)\ldots.\lambda^{x_1}_{\mu_s}(x_3\,x_2) - traces,$$

$$\lambda^{x_1}_{\mu}(x_3\,x_2) = \frac{(x_{12})_\mu}{x_{12}^2} - \frac{\overline{(x_{13})_\mu}}{x_{13}^2}$$

where $\{\ldots\}$ denotes quasilocal terms that contain derivatives of $\delta(x_{13})$. They are too cumbersome for being presented here. Two other functions at $l = d + s$ are obtained by calculating the residue of $C^{l,s}_{1,2}$. We denote them as:

$$C^{s}_{1\mu\nu} = \underset{l=d+s}{res}\, C^{ls}_{1\mu\nu},\quad C^{s}_{2\mu\nu} = \underset{l=d+s}{res}\, C^{ls}_{2\mu\nu} \tag{4.13}$$

Explicit expressions for them for $s \geqslant 2$ are very tedious, being simple only for $s = 1$:

$$C_{1\mu\nu\rho} \sim (x_{12}^2)^{-d}\Big[\delta_{\mu\rho}\partial_\nu^{x_3}\ \delta(x_{13}) +$$

$$+\ 2D\,\delta_{\nu\rho}\,\frac{(x_{12})_\mu}{x_{12}^2}\ \delta(x_{13}) + (\mu \leftrightarrow \nu) - trace\Big]$$

Therefore, the general expression for the Green function $\langle P_s^{(i)}\ \varphi T_{\mu\nu}\rangle$ can be written in the form $(s \geqslant 2)$

(4.14) $$\langle P_s^{(i)}\ \varphi T_{\mu\nu}\rangle = g_i\, C_{\mu\nu}^{reg} + \alpha_1^{(i)}\, C_{1\mu\nu}^s + \alpha_2^{(i)}\, C_{2\mu\nu}^s$$

where g_i, $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ are constants. The important feature of these functions is that they have the same (up to a factor) power part $C_{\mu\nu}^{reg}$.

4. We are going to show now that only two, out of the three, fields $P_s^{(i)}$ can contribute into expansion of the product $\varphi(x)\,\chi(x+\epsilon)$. Let us pass now to the new fields

$$P_s^{(i)} \to \sum_{k=1}^{3} \alpha_k^i\, P_s^{(k)}$$

preserving the conditions (4.9) and (4.9a). The new fields should be chosen so that one of the three (at a given s) Green functions $\langle P_s^{(i)}\ \varphi T_{\mu\nu}\rangle$ consist of quasilocal terms only (4.13):

(4.15) $$\langle P_s^{(1)}\ \varphi T_{\mu\nu}\rangle = \alpha_1\, C_{1\mu\nu}^s + \alpha_2\, C_{2\mu\nu}^s$$

The corresponding Wightmann function in the Minkovski space is equal to zero:

(4.16) $$\langle 0\,|\,P_s^{(1)}\ \varphi T_{\mu\nu}\,|\,0\rangle_{Mink} = 0$$

It can be assumed, thus, that the field $P_s^{(1)}$ is absent: it is zero as an operator. Then it is possible that

(4.17) $$\langle P_s^{(1)}\ \varphi\chi\rangle = 0,\ \langle P_s^{(1)}\ \varphi T_{\mu\nu}\rangle = \text{quasilocal terms.}$$

Later this assumption should be verified by studying the function $\langle\varphi\chi\varphi\chi\rangle$. Condition (4.17) means that the pole corresponding to the field $P_s^{(1)}$ is absent from its partial wave expansion, although there is such a pole in the first term of the expansion (4.6). Note, that the operator equality $P_s^{(1)} = 0$ means, besides (4.17), that $\langle\varphi\varphi\varphi P_s^{(1)}\rangle = 0$, etc. For our purposes, only the condition $\langle P_s^{(1)}\ \varphi\chi\rangle = 0$ should be verified, since it ensures that there is no contribution of the field $P_s^{(1)}$ to the operator product $\varphi(x)\,\chi(x+\epsilon)$.

The quasilocal terms (4.15) are needed in the theory even if $P_s^{(1)} \equiv 0$. Their sense will be discussed in the following Section.

Therefore, the operator equation (2.1) as a matter of fact implies that

$$P_s^{(2)} = P_s^{(3)} = 0 \tag{4.18}$$

V. COMPLETE CONTRIBUTION OF THE POLES IN $l = d + s$

1. Let us consider the Green function $\langle \varphi\chi\varphi T_{\mu\nu}\rangle$. Its partial wave expansion can be written in the form:

(5.1) $\langle \varphi\chi\varphi T_{\mu\nu}\rangle = \sum_s \int dl \;\; \rho(l, s)$  $+$

$+ \sum_s \int dl \, \rho^{tr}(l, s)$ 

where

$$C^{l,s}_{\mu\nu} = A_1(l, s)\, C^{l,s}_{1\mu\nu} + A_2(l, s)\, C^{l,s}_{2\mu\nu} \tag{5.2}$$

The function $A_{1,2}(l, s)$ and $\rho(l, s)$ are derived [2, 10] from the Ward identity,

$$C^{tr}_{\mu\nu} = C^{reg}_{\mu\nu} + B_1(l, s)\, C^{ls}_{1\mu\nu} + B_2(l, s)\, C^{ls}_{2\mu\nu} \tag{5.3}$$

The coefficients $B_{1,2}(l, s)$ are derived from the transversality condition $\partial_\mu C^{tr}_{\mu\nu} = 0$. We have [8]:

$$B_1(l, s) \sim B_2(l, s) \sim \frac{1}{\Gamma\left(\frac{l - s - d}{2}\right)} \tag{5.4}$$

i.e. $C^{tr}_{\mu\nu}$ differ from $C^{reg}_{\mu\nu}$, see (4.12a), by quasilocal terms only.

The complete contribution of the poles in the point $l = d + s$ can be determined from (4.4). It is equal to

$$\langle \varphi(x)\, \chi(x + \epsilon)\, \varphi(x_1)\, T_{\mu\nu}(x_2)\rangle\big|_{\epsilon \to 0} \sim \tag{5.5}$$

$$\sim \frac{1}{(\epsilon^2)^{\frac{\Delta - s}{2}}} \left\{ (\rho_1\, C^s_{1\mu\nu} + \rho_2\, C^s_{2\mu\nu}) + \underset{l = d + s}{res}\; \rho^{tr}\, C^{tr}_{\mu\nu} \right\}$$

The coefficients $\rho_{1,2} \equiv \rho_{1,2}(d, s)$ are determined by the Ward identity. The right-hand side is the common contribution of the fields $P_s^{(2)}$, $P_s^{(3)}$ and quasilocal terms (4.15):

$$\rho_1 C^s_{1\mu\nu} + \rho_2 C^s_{2\mu\nu} + \underset{l=d+s}{res}\ \rho^{tr} C^{tr}_{\mu\nu} = \tag{5.6}$$
$$= \alpha_1 C^s_{1\mu\nu} + \alpha_2 C^s_{2\mu\nu} + \langle P^{(2)}_s \varphi T_{\mu\nu}\rangle + \langle P^{(3)}_s \varphi T_{\mu\nu}\rangle$$

Here we took into account that with regard for (5.3), (5.4) and (4.13) one can write

$$\langle P^{(2)}_s \varphi T_{\mu\nu}\rangle + \langle P^{(3)}_s \varphi T_{\mu\nu}\rangle = \tag{5.7}$$
$$= \beta_1 C^s_{1\mu\nu} + \beta_2 C^s_{2\mu\nu} + \underset{l=d+s}{res}\ \rho^{tr} C^{tr}_{\mu\nu}$$

Thus, the Ward identities fix the sum of terms of different nature: a part of the common contribution of the fields $P^{(2)}_s$, $P^{(3)}_s$ (the terms $\beta_1 C^s_{1\mu\nu} + \beta_2 C^s_{2\mu\nu}$) and the contribution of quasilocal terms (4.15). We have

$$\alpha_1 + \beta_1 = \rho_1(d, s), \quad \alpha_2 + \beta_2 = \rho_2(d, s) \tag{5.8}$$

Assuming, in accord with (4.18), that $P^{(2)}_s = P^{(3)}_s = 0$ for a certain s, we shall obtain a number of consequences in the sections below. One of them is that $\beta_1 = \beta_2 = 0$, and in (5.5) only the contribution of the quasilocal terms (with the same s) is preserved:

$$\langle \varphi(x)\, \chi(x+\epsilon)\, \varphi T_{\mu\nu}\rangle |_{\epsilon\to 0} \sim \frac{1}{(\epsilon^2)^{\frac{\Delta - s}{2}}} (\alpha_1 C^s_{1\mu\nu} + \alpha_2 C^s_{2\mu\nu}) \tag{5.9}$$

where α_1 and α_2 are fixed by the Ward identity. Note that these terms are needed in the theory. If we demand that they be absent (i.e. connect them to the field $P^{(1)}_s$ and assume that $P^{(1)}_s = 0$) then the contradiction to the Ward identity appears. The meaning of these terms is that they contribute to the field equation for the Green function $\langle \varphi\varphi T_{\mu\nu}\rangle$. These are just the quasilocal terms which appear from the commutator $[\varphi, T_{0\mu}]$ when calculating the derivatives $\square$ and $\square^2$ of the T-ordered product of the fields $\varphi T_{\mu\nu}$. A detailed description one can find in [10] where the Green function $\langle \varphi\varphi^+ j_\mu\rangle$ containing the current is taken as example. The corresponding terms there appear from the commutator $[\varphi, j_0]$ when calculating the derivatives $\square \langle \varphi\varphi j_\mu\rangle$ in the field equation.

2. Let us find the solution of the Ward identities for arbitrary Green functions $G^{(n)}_{\mu\nu} = \langle T_{\mu\nu}\ \varphi(x_1), \ldots, \varphi(x_n)\rangle$. We are going to show that two out of the three kernels $G^{(i)}_{ls}$ in (4.6) are uniquelly determined by the Ward identity, see (5.13). We take advantage of the fact that $\partial_\mu T_{\mu\nu}$ is transformed [9] as a con-

formal vector of dimensional $D + 1$. The quantity $\partial_\mu G^{(n)}_{\mu\nu}$ is completely determined by the Ward identity and, thus, it is possible to find the kernels of its partial wave expansion. It involves two invariant three-point functions of the form $V^{ls}_\mu = \partial_\nu \langle P^l_s \varphi T_{\mu\nu} \rangle$. It is convenient to choose in (4.6)

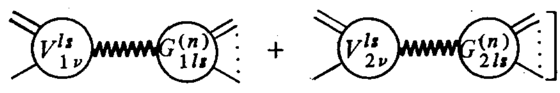
$$C^l_{3\mu\nu} = C^{tr}_{\mu\nu}$$

Then the two invariant functions $V^{ls}_{1\mu}$ and $V^{ls}_{2\mu}$ in the expansion of $\partial_\mu G^{(n)}_{\mu\nu}$ are equal to:

(5.10) $$V^{ls}_{1\mu} = \partial_\mu C^{ls}_{1\mu\nu}, \qquad V^{ls}_{2\mu} = \partial_\nu C^{ls}_{2\mu\nu}$$

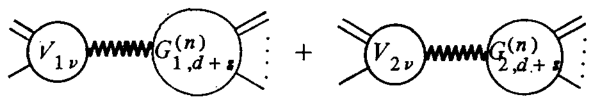
Their forms are given in [8]. We have:

(5.11) $$\partial_\mu G^{(n)}_{\mu\nu} = \sum_s \int dl$$



The kernels $G^{(n)}_{1ls}$ and $G^{(n)}_{2ls}$ are unambiguously determined in terms of the quantity $\partial_\mu G^{(n)}_{\mu\nu}$, see Sections VI, VII, that in known from the Ward identity. The functions $V^{ls}_{1\nu}$ and $V^{ls}_{2\nu}$ have poles in the points $l = d + s$. The total residue is equal to

(5.11a) $$\operatorname*{res}_{l=d+s} \partial_\mu G^{(n)}_{\mu\nu} =$$



where

$$V_{i\nu} = \operatorname*{res}_{l=d+s} V^{ls}_{i\nu}, \quad G^{(n)}_{i,d+s} = \tilde{G}^{(n)}_{ils}\big|_{l=d+s}; \; i = 1, 2$$

The partial wave expansion of the Green function $G^{(n)}_{\mu\nu}$ can be easily restored from (5.11). When going from $\partial_\mu G^{(n)}_{\mu\nu}$ to $G^{(n)}_{\mu\nu}$ the following replacement should be done

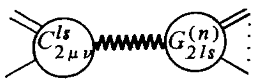
(5.12) $$V^{ls}_{1\nu} \to C'^{ls}_{1\mu\nu} = C^{ls}_{1\mu\nu} + A(l, s)\, C^{tr}_{\mu\nu},$$

$$V^{ls}_{2\nu} \to C'^{ls}_{2\mu\nu} = B(l, s)\, C^{tr}_{\mu\nu} + \frac{1}{\Gamma(l - d - s)} C^{ls}_{2\mu\nu}$$

and a trasverse term added. Whatever the unknown coefficients $A(l, s)$ and $B(l, s)$ in (5.12) are, one can always pass over to such combinations of the functions (5.12) that $C'^{ls}_{1\mu\nu}$ satisfy the condition (5.2). So, we have for the Green function

(5.13) $G^{(n)}_{\mu\nu} = \sum_s \int dl \Big[$ 

$+$

$+$ 

$+$ 

$\Big]$

This is the general solution of the Ward identity. It has the following features:

1. The kernels $\tilde{G}^{(n)}_{1ls}$ and $G^{(n)}_{2ls} = 1/(l-d-s)\,\tilde{G}^{(n)}_{2ls}$ are known from the Ward identities;

2. $\operatorname*{res}_{l=d+s} C^{ls}_{\mu\nu} = \alpha_1\, C^{s}_{1\mu\nu} + \alpha_2\, C^{s}_{2\mu\nu} =$ quasilocal terms;

3. $C'^{ls}_{2\mu\nu}\big|_{l=d+s} = \beta_1\, C^{s}_{1\mu\nu} + \beta_2\, C^{s}_{2\mu\nu} + B'(d,s)\, C^{1r}_{\mu\nu}$ (5.14)

where the constants $\alpha_{1,2}$ and $\beta_{1,2}$ are known from the Ward identity.

4. The kernel $G^{(n)}_{2ls}$ has poles in the points $l = d + s$.

5. The third term in (5.13) is transversal.

6. The function $B'(d, s)$ in (5.14) and the kernel $G^{(n)}_{3ls}$ in the third term have remained unknown quantities.

7. $\operatorname*{res}_{l=d+s} G^{(n)}_{2ls} = \langle P^{(2)}_s\, \varphi \ldots \varphi \rangle$

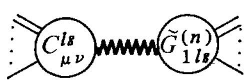
VI. CONDITION FOR THE ABSENCE OF THE FIELDS P_s

Our task now is to study the consequences of the equations

(6.1) $$\langle P^{(2)}_s \varphi_1 \ldots \varphi_n \rangle = \langle P^{(3)}_s \varphi_1 \ldots \varphi_n \rangle = 0$$

To this end let us express the Green functions $\langle P^{(i)}_s\, \varphi_1, \ldots, \varphi_n \rangle$ with $i = 2, 3$ in terms of the kernel of the partial wave expansion (4.6) of the Green function $G^{(n)}_{\mu\nu} = \langle T_{\mu\nu}\, \varphi_1, \ldots, \varphi_n \rangle$, see (6.10) below.

The contribution of all the poles in the point $l = d + s$ to the Green function $G^{(n)}_{\mu\nu}$ can be written in the form (see (4.2) and (4.6)):

(6.2) $\operatorname*{res}_{l=d+s} G^{(n)}_{\mu\nu} = \operatorname*{res}_{l=d+s}$ 

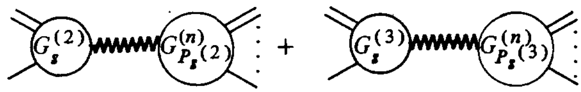
$+ G^{(n)}_{\mu\nu}\big|_{P_s}$

Here the function $C^{ls}_{\mu\nu}$ is chosen so that its residue in $l = d + s$ coincide with the quasilocal term in (5.9):

(6.3) $$\underset{l=d+s}{\mathrm{res}}\ C^{ls}_{\mu\nu} = \alpha_1\, C^{s}_{1\mu\nu} + \alpha_2\, C^{s}_{2\mu\nu}$$

while $G^{(n)}_{\mu\nu}|_{P_s}$ is the contribution of the fields $P^{(2)}_s$ and $P^{(3)}_s$ together

(6.4) $$G^{(n)}_{\mu\nu}|_{P_s} =$$



The green functions $G^{(2)}_s$ and $G^{(3)}_s$ can be written in the form

(6.5) $$G^{(2)}_s = \langle P^{(2)}_s\, \varphi T_{\mu\nu}\rangle = g_2 C^{reg}_{\mu\nu} + \alpha^{(2)}_1 C^{s}_{1\mu\nu} + \alpha^{(2)}_2 C^{s}_{2\mu\nu}$$

(6.6) $$G^{(3)}_s = \langle P^{(3)}_s \varphi T_{\mu\nu}\rangle = g_3\, C^{reg}_{\mu\nu} + \alpha^{(3)}_1 C^{s}_{1\mu\nu} + \alpha^{(3)}_2\, C^{s}_{2\mu\nu}$$

As assumed in eqs. (6.2) and (6.4) the invariant functions $C^{ls}_{i\mu\nu}$ in (4.6) are chosen as follows:

(6.7) $$C^{ls}_{1\mu\nu} = C^{ls}_{\mu\nu},\ C^{ls}_{2\mu\nu}\,|_{\,l=d+s} = G^{(2)}_s,\ C^{ls}_{3\mu\nu}\,|_{\,l=d+s} = G^{(3)}_s$$

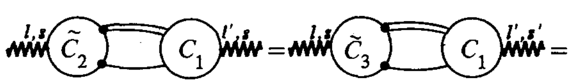
The Green functions then

(6.8) $$G^{(n)}_{P_s(2)} = \langle P^{(2)}_s \varphi_2 \ldots \varphi_n\rangle,\ G^{(n)}_{P_s(3)} = \langle P^{(3)}_s\, \varphi_2 \ldots \varphi_n\rangle$$

are derived as residues of the kernels $G^{(2)}_{ls}$ and $G^{(3)}_{ls}$ in expansion (4.6).

Let us express these kernels in term of the Green function $G^{(n)}_{\mu\nu}$. For doing this, similarly to [1,2] we introduce three more invariant 3-point functions of the type of $\langle P^{sl}_s\, \tilde\varphi\, \tilde T_{\mu\nu}\rangle$ where $\tilde\varphi$ is a scalar of dimension $D - d$, and $\tilde T_{\mu\nu}$ is a tensor of zero dimension. Their notations are $\tilde C^{ls}_{1\mu\nu}$, $\tilde C^{ls}_{2\mu\nu}$ and $\tilde C^{ls}_{3\mu\nu}$. These functions should be selected so that the orthogonality conditions be fulfilled

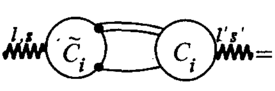
(6.9)



$= 0$

etc. Normalization of the functions C^{ls}_i is established by the condition

(6.10)



$= I, \qquad i = 1, 2, 3$

The notation in (6.9) and (6.10) are the same as in [1, 2]. The dot on the line means that it has been already amputated; therefore, here it is δ-functions and not propagators that are put in correspondence with the internal lines.

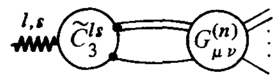
Now we have for the kernel $G^{(i)}_{ls}$ in (4.6)

(6.11) $G^{(i)}_{ls} =$ 

More detailed description of relations (6.9) - (6.11) will be given elsewhere. A more simple case of the current containing functions is considered in [1, 2].

Using equations (6.11) we obtain two conditions for each Green function $G^{(n)}_{\mu\nu}$:

(6.12) $\underset{l=d+s}{\mathrm{res}}$  $= 0$

(6.13) $\underset{l=d+s}{\mathrm{res}}$  $= 0$

for the given s.

Equations (6.12) and (6.13) represent a sufficient condition that ensures the absence of the fields P_s. This statement refers to the field theories at $D > 2$. There is solely equation (6.12) in the two-dimensional space, see Section X. In this case (6.12) is necessary and sufficient condition for the field P_s to be absent.

VII. DIFFERENTIAL EQUATIONS FOR THE GREEN FUNCTIONS

Let us transform the conditions (6.12) and (6.13) to a more suitable form. Introduce new fields

$$P'^{(i)}_s = \sum_{k=1}^{3} \alpha^i_k \, P^{(k)}_s , \qquad i = 1, 2, 3$$

so that the orthonormalization conditions (4.9), (4.9a) be preserved.

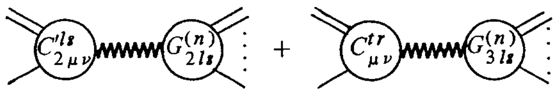
The coefficients α^i_k form a three-parameter matrix family of the group SO(3). There is a unique choice of parameters with which, firstly, the function $\langle P'^{(1)}_s \; \varphi \; T_{\mu\nu} \rangle$ remains quasilocal, see (4.15) and, secondly, the Green function $\langle P'^{(3)}_s \; \varphi \; T_{\mu\nu} \rangle$ transversal:

(7.1) $$\partial^{x_3}_{\mu} \langle P^{(3)}_s(x_1)\, \varphi(x_2)\, T_{\mu\nu}(x_3) \rangle = 0$$

Here and below the primes for notation of the new fields are admitted.

The choice of the fields is unambiguously set by these conditions. With such a choice of the fields it is convenient to use partial wave expansion of the Green function $G^{(n)}_{\mu\nu}$ in the form (5.13). Residues of the kernel $G^{(n)}_{2ls}$ and $G^{(n)}_{3ls}$ in the pole $l = d + s$ are explicitly expressed through the Green functions $G^{(n)}_{P_s}(2)$ and $G^{(n)}_{P_s}(3)$ of the original fields. The total contribution of the fields can be now rewritten in the form

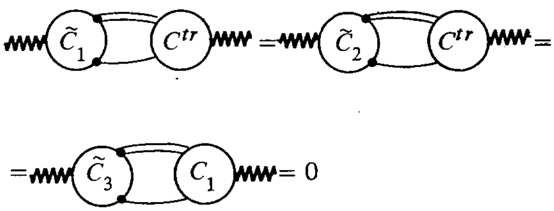
(7.2) $\qquad G^{(n)}_{\mu\nu}\,|_{P_s} = \underset{l=d+s}{\mathrm{res}}$



Unlike (6.3), the second term here is transversal

In addition to (6.9) let us demand that

(7.3)



i.e. that $\tilde{C}_1$ and $\tilde{C}_2$ be orthogonal to the transversal function. As it is shown in [8, 10], such functions $\tilde{C}$ are "longitudinal". For $\tilde{C}_{2\mu\nu}$, particularly, we have [7, 8].

(7.4)
$$\tilde{C}_{2\mu\nu}(x_1\, x_2\, x_3) = \langle P^l_{\mu_1 \ldots \mu_s}(x_1)\, \tilde{\varphi}(x_2)\, \tilde{T}_{\mu\nu}(x_3)\rangle =$$
$$= \partial^{x_3}_{\mu} B_{\nu,\mu_1\ldots\mu_s} + \partial^{x_3}_{\nu} B_{\mu,\mu_1\ldots\mu_s} - \frac{2}{D}\,\delta_{\mu\nu}\partial^{x_3}_{\lambda} B_{\lambda,\mu_1\ldots\mu_s}$$

The orthogonality conditions (7.3) are fulfilled identically if $\tilde{C}_1$ and $\tilde{C}_2$ are substituted in this form. The functions $B_{\mu,\mu_1\ldots\mu_s}$ are equal [8] to:

(7.5)
$$B_{\mu,\mu_1\ldots\mu_s}(x_1\, x_2\, x_3) = B_{1\mu,\mu_1\ldots\mu_s}(x_1\, x_2\, x_3) + \tilde{\alpha} B_{2\mu,\mu_1\ldots\mu_s}(x_1 x_2 x_3)$$
$$B_{1\mu,\mu_1\ldots\mu_s} = \lambda^{x_3}_{\mu}(x_1 x_2)\, \lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3 x_2)\, \Delta(x_1\, x_2\, x_3)$$
$$B_{2\mu,\mu_1\ldots\mu_s} = \frac{1}{x^2_{13}} \sum_{k=1}^{s} g_{\mu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3\, x_2)\, \Delta(x_1\, x_2\, x_3)$$
$$\Delta(x_1\, x_2\, x_3) = (x^2_{13})^{-\frac{l+d-D-d-2}{2}}\,(x^2_{23})^{\frac{l+d-D-s+2}{2}}$$
$$(x^2_{12})^{-\frac{l-d+D-s+2}{2}}$$

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - 2\,\frac{x_\mu\, x_\nu}{x^2} \quad \text{and} \quad \lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3\, x_2)$$

are given in (4.12a). There is an unknown constant $\bar{\alpha}$ in (7.5). It will be determined later. The function $\tilde{C}_{1\mu\nu}$ differs from $\tilde{C}_{2\mu\nu}$ by this constant value.

Let us again consider the conditions (6.12) and (6.13). With our choice of the functions $\tilde{C}_{2\mu\nu}$ and $\tilde{C}_{3\mu\nu}$ one of these conditions is equivalent to the requirement that the first term in (7.2) be absent, while the other implies the absence of the second term. Let us consider the first condition. We are going to demonstrate that it is equivalent to the differential equation for the Green function

$$G(x_1\, x_2\, \ldots x_n) = \langle \varphi_1(x_1)\, \varphi_2(x_2) \ldots \varphi_n(x_n)\rangle.$$

Really,

$$\underset{l=d+s}{\mathrm{res}} \int dy_1\, dy_2\, \tilde{C}_{2\mu\nu,\mu_1\ldots\mu_s}(x_1\, y_1 y_2)$$

$$\langle \varphi_1(y_1)\, T_{\mu\nu}(y_2)\varphi_2(x_2) \ldots \varphi_n(x_n)\rangle =$$

$$= -2 \underset{l=d+s}{\mathrm{res}} \int dy_1\, dy_2\, B_{\mu,\mu_1\ldots\mu_s}(x_1 y_1 y_2)\, \partial^{y_2}_\nu$$

$$\langle \varphi_1(y_1)\, T_{\mu\nu}(y_2)\, \varphi_2(x_2) \ldots \varphi_n(x_n)\rangle.$$

Let us substitute here $\partial^{y_2}_\nu \langle \varphi_1\, T_{\mu\nu}\varphi_2 \ldots \varphi_n\rangle$ using the Ward identity:

$$\partial^{y_2}_\nu \langle \varphi_1\, T_{\mu\nu}\varphi_2 \ldots \varphi_n\rangle = -\left[\sum_i \delta(y_2 - x_i)\partial^{x_i}_\nu + \ldots\right]$$

$$\langle \varphi_1(y_1)\varphi_2(x_2) \ldots \varphi_n(x_n)\rangle.$$

The integral over y_2 can be calculated due to the δ-functions (the term proportional to $\delta(y_1 - y_2)$ makes zero contribution). The remaining integral over y_1 can be also calculated. Let us introduce $\underset{l=d+s}{\mathrm{res}}$ under the integral sign, and then determine the residue of the expression in the integrand using relation (4.10). As a result, subject to the integration over y_1 is the sum of terms with derivatives of $\delta(y_1 - x_i)$ up to the order $(s + 2)$. It can be shown that the derivatives of higher order cancel out.

Calculating now

$$\int dy_1$$

we obtain the differential equation

$$L^{(s)}\left(x_1 \ldots x_n, \frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_n}\right) \langle \varphi_1(x_1) \ldots \varphi_n(x_n) \rangle = 0 \tag{7.6}$$

where the differential operator $L^{(s)}$ depends on the tensor structure of the fields φ_i and on the form of the Ward identities. In the general case this equation is extremely cumbersome and will be given in another paper. The calculations become essentially simpler in the two-dimensional space. At $D = 2$ and $s = 2$ they give [6, 7]:

$$\left\{\frac{3}{2(d+1)} (\partial_{\pm}^{x_1})^2 - \sum_{i=2}^{y} \frac{1}{x_{1i}^{\pm}} \partial_{\pm}^{x_i} - \frac{d}{2} (x_{13}^{\pm})^{-2} - \right.$$
$$\left. - \frac{\Delta}{2} [(x_{12}^{\pm})^{-2} + (x_{14}^{\pm})^{-2}]\right\} \tag{7.7}$$
$$\langle \varphi(x_1)\, \chi(x_2)\, \varphi(x_3)\, \chi(x_4) \rangle = 0$$

where $x_{\pm} = x_1 \pm i\, x_2$ are the light cone variables. This coincides with the equation in [4] for the two-dimensional Ising model. The two-dimensional models for all s are studied in [12]. It is shown, in particular, that at $D = 2$ and $s = 2$, 3 equations (7.6) coincide with the corresponding equations in the minimal models in paper [4], see also [13] and the reference therein, whereas at $s \geqslant 4$ they differ from them.

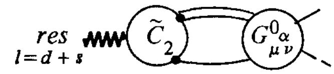
Note that the differential operator $L^{(s)}$ in (7.6) depends on the parameter $\bar{\alpha}$ involved in the function (7.5). This parameter is calculated together with the field dimensions and other parameters of the theory, see Section IX. In particular, when deducing equation (7.7), it was taken into account that $\bar{\alpha} = 1/4$ for $D = 2$, $s = 2$. The third derivatives, at this value of $\bar{\alpha}$, are cancelled from (7.7).

After differential equations (7.6) have been solved one may address the condition (6.13). The only thing to do now is to verify if it is fulfilled, after of the solution of equations (7.6) is substituted into it for the case when one of the fields φ_i is the energy-momentum tensor. In particular, for the Green function $\langle \varphi\, \chi\, \varphi\, T_{\mu\nu} \rangle$ determined from (7.6) the condition (6.13) will mean that at a given s the pole in $l = d + s$ in absent from the second term of expansion (5.1).

There remains one more condition of selfconsistency mentioned at the end of Section IV, which provides the absence of the contribution into $\varphi(x)\chi(x+\epsilon)$, from the field $P_s^{(1)}$ see (4.17). To check it we should consider the partial wave expansion of the function $\langle \varphi\, \chi\, \varphi\, \chi \rangle$, derived from (7.6) and see that at a given s the pole in $l = d + s$ is absent. This condition is identically fulfilled in two-dimensional models.

VIII CALCULATION OF SCALE DIMENSIONS OF FIELDS

Dimension d_α of any composite field O_α is determined from the condition (6.12) written for the Green function $\langle \varphi T_{\mu\nu}\, \varphi O_\alpha \rangle$. We have:

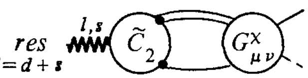


$= 0$

In particular, there are two scalar composite fields: χ and P^{D-2}. The field P^{D-2} has been already mentioned in Section II, see also Section IX. Its dimension is known: $d_P = D - 2$. Let us consider condition (6.12) for the Green functions

$$G^{\chi}_{\mu\nu} = \langle \varphi\, \chi\, \varphi\, T_{\mu\nu} \rangle \quad \text{and} \quad G^{P}_{\mu\nu} = \langle \varphi\, P^{D-2}\, \varphi T_{\mu\nu} \rangle .$$

We have:



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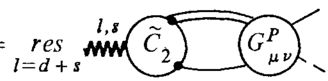


$= 0$

These equations produce two equations for dimension d and Δ of the fields φ and χ:

$$f_1^{(s)}(d, \Delta) + \bar{\alpha} f_2^{(s)}(d, \Delta) = 0, \tag{8.1}$$

$$f_1^{(s)}(d, D-2) + \bar{\alpha} f_2^{(s)}(d, D-2) = 0$$

where $f_1^{(s)}$ and $f_2^{(s)}$ are functions (2.6), and $\bar{\alpha}$ is the parameter involved in (7.5). Thus, parameter $\bar{\alpha}$ is still to be determined.

For doing this we consider equation (6.12) in the case of the Green function $G_T = \langle \varphi T_{\mu\nu}\, \varphi T_{\rho\sigma} \rangle$. Its partial wave expansion requires special discussion, because formally it has poles of the second order in the points $l = d + s$. They also can be found in every term of the expression under the sign of $\underset{l=d+s}{\mathrm{res}}$ in equation (6.12). We have:

(8.2) 

$$= b_1(l,s)\, C^{ls}_{1\mu\nu} + b_2(l,s)\, C^{ls}_{2\mu\nu} + b_3(l,s)\, C^{ls}_{3\mu\nu}$$

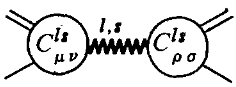
where the functions $b_i(l, s)$ are derived from the Ward identity (9.1). It can be shown that they have poles in the points $l = d + s$:

$$b_1(l,s) \sim b_2(l,s) \sim b_3(l,s) \sim \Gamma\left(\frac{l-d-s}{2}\right)$$

The requirement that the fields $P^{(2)}_s$ and $P^{(3)}_s$ be absent is, in this case, equivalent to the three algebraic equations

$$\underset{l=d+s}{\mathrm{res}}\, b_1(l,s) = \underset{l=d+s}{\mathrm{res}}\, b_2(l,s) = \underset{l=d+s}{\mathrm{res}}\, b_3(l,s) = 0 \tag{8.3}$$

Substantiation of this statement requires elucidation of some nice technical details (*) that are not presented here. We only notice that if (8.3) is fulfilled at a certain s, then only one pole term remains in the partial wave expansion in the vicinity of $l = d + s$:

(8.4) $\rho^{reg}(l, s)$ 

where $\rho^{reg}(l, s)$ is regular, and $C^{ls}_{\mu\nu}$ is a singular function determined in (6.7) and (6.5). Although formally the term (8.4) has a pole of the second order, in fact this term leads to the first-order pole in the expansion of the Green function $\langle \varphi(x + \epsilon)\, T_{\mu\nu}(x)\, \varphi T_{\mu\nu} \rangle$ with respecto to ϵ, because the function $\tilde{Q}^{ls}_{\mu\nu}$ analogous to (4.4) is regular. The residue in the pole is proportional to quasi-local term (5.9). Conditions (8.3) provide the absence of other contributions.

So, the condition for the fields $P^{(2,\,3)}_s$ to be absent in the case of the Green function G_T is equivalent to three algebraic equations (8.3). They determine the parameter $\tilde{\alpha}$ together with two additional parameters involved in the anomalous Ward idendity for $\langle \varphi\, T_{\mu\nu}, \varphi T_{\rho\sigma} \rangle$.

(*) It can be shown that conditions (8.3) provide cancelling of poles of the second order in the sum of terms in the right-hand side of equation (8.2). (4.11) is used in the proof.

IX. ANOMALOUS WARD IDENTITY

The most general form of the Ward identity admitted by conformal invariance involves anomalous terms of two types (at $D > 2$)

$$
\begin{aligned}
&\partial_\mu^{x_1} \langle T_{\mu\nu}(x_1)\, T_{\rho\sigma}(x_2)\, \varphi(x_3)\, \varphi(x_4)\rangle = \\
&= -\Big\{ \delta(x_{13})\, \partial_\nu^{x_3} + \delta(x_{14})\, \partial_\nu^{x_4} - \\
&- \frac{d}{D}\, \partial_\nu^{x_1}\, [\delta(x_{13}) + \delta(x_{14})] + \\
&+ \delta(x_{12})\, \partial_\nu^{x_2} - 2a_1\, \partial_\nu^{x_1}\, \delta(x_{12}) \Big\} \langle T_{\rho\sigma}(x_2)\, \varphi(x_3)\, \varphi(x_4)\rangle \\
&+ 2\{a_2\, \partial_\rho^{x_1}\, \delta(x_{12})\, \langle T_{\nu\sigma}(x_2)\, \varphi(x_3)\, \varphi(x_4)\rangle + \\
&+ a_3\, \partial_\tau^{x_1}\, \delta(x_{12})\, \delta_{\nu\rho} \langle T_{\tau\sigma} \varphi\varphi\rangle + (\rho \leftrightarrow \sigma) - trace\} \\
&+ \{C_1\, \partial_\nu^{x_1}\, \delta(x_{12})\, \partial_\rho^{x_2}\, \partial_\sigma^{x_2} + 2C_2\, \partial_\rho^{x_1}\, \delta(x_{12})\, \partial_\nu^{x_2}\, \partial_\sigma^{x_2} + \\
&+ 2C_3\, \partial_\tau^{x_1}\, \delta(x_{12})\, \delta_{\nu\rho}\, \partial_\sigma^{x_2}\, \partial_\tau^{x_2} + e_1\, \partial_\rho^{x_1}\, \partial_\sigma^{x_1}\, \delta(x_{12})\, \partial_\nu^{x_2} + \\
&+ 2e_2\, \partial_\nu^{x_1}\, \partial_\sigma^{x_1}\, \delta(x_{12})\, \partial_\rho^{x_2} + 2e_3\, \delta_{\nu\rho}\, \partial_\sigma^{x_1}\, \partial_\tau^{x_1}\, \delta(x_{12})\, \partial_\tau^{x_2} \\
&+ e_4\, \Box\delta(x_{12})\, \delta_{\nu\rho}\, \partial_\sigma^{x_2} + 3f_1\, \partial_\nu^{x_1}\, \partial_\rho^{x_1}\, \partial_\sigma^{x_1}\, \delta(x_{12}) + \\
&+ 3f_2\, \delta_{\nu\rho}\, \partial_\sigma^{x_1}\, \Box\, \delta(x_{12}) \\
&+ (\rho \leftrightarrow \sigma) - trace\} \langle P^{D-2}(x_2)\, \varphi(x_3)\, \varphi(x_4)\rangle,
\end{aligned}
\tag{9.1}
$$

where $\langle D^{D-2}\, \varphi\varphi\rangle$ is the Green function (2.4)

$$
\begin{aligned}
&Da_1 + 2a_2 + 2a_3 = \frac{D}{2}\,, \quad 2a_2 - 2a_3 = 1 \\
&C_1 = \frac{6}{(D-1)(D-2)}\, f_2, \\
&C_2 = -\frac{3D}{2(D-1)(D-2)}\, f_2, \qquad C_3 = C_2 \\
&e_1 = \frac{3D}{(D-1)(D-2)}\, f_2, \\
&e_2 = \frac{3(D^2 - 4D + 2)}{D(D-1)(D-2)}\, f_2,
\end{aligned}
\tag{9.2}
$$

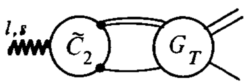
$$e_3 = -\frac{3D}{2(D-2)} f_2,$$

$$e_4 = \frac{6}{(D-2)} f_2, \qquad f_1 = -\frac{3D-2}{D(D-1)} f_2.$$

The calculations are given in our (with V.N. Zaikin as a co-author) paper [12].

In fact, one may put $f_2 = 1$, because an unknown normalizing factor C is involved in the Green function $\langle P^{D-2} \varphi\varphi \rangle$, see (2.4). Therefore, there are two unknow parameters. Let them be a_1 and C.

The functions b_1, b_2 and b_3 in (8.2) are derived from the Ward identity and expressed through these parameters. Thus, the equation

(9.3) $\quad \operatorname*{res}_{l=d+s}$  $= 0$

is equivalent to the three algebraic equations for three parameters

(9.4) $\quad \bar{\alpha}, \ a_1, \ C.$

From the abovesaid it follows that the anomalous terms in the Ward identity are needed due to the structure of the theory, because without them the number of equations would exceed that of the parameters.

X. TWO-DIMENSIONAL THEORIES

Setting $D = 2$ in all equations we obtain an infinite family of two-dimensional solvable models. All the calculations become much easier at $D = 2$ because:

1. There are only two independent invariant functions of the type of $\langle P^l_s \varphi T_{\mu\nu} \rangle$. Any asymmetrical traceless tensor $T_{\mu\nu}$ can be represented in the form: $T_{\mu\nu} = \partial_\mu T_\nu + \partial_\nu T_\mu - \delta_{\mu\nu}\partial_\lambda T_\lambda$ therefore, any conformal-invariant function is representable in the form (7.4).

2. There is no transversal function $C^{tr}_{\mu\nu}$; thus, instead of two fields $P^{(2)}_s$ and $P^{(3)}_s$ only one field $P^{(2)}_s \equiv P_s$ is available. Equation (6.13) becomes unnecessary, and (6.12) is still equivalent to the differential equation, see (7.7). All calculations of Section VII can be formally transferred to the case of $D = 2$. The calculations can be easily carried out in the light cone variables.

3. The anomalous Ward identity essentially simplifies: the terms $\sim a_i$

disappear; the field P^{D-2} is constant; and out of the last nine terms in (9.1) only two terms differ from zero; also, the third term is absent from (8.2). Thus, at $D = 2$ equation (9.3) is equivalent to the two algebraic equations

$$\operatorname*{res}_{l=d+s} b_1 = \operatorname*{res}_{l=d+s} b_2 = 0$$

for the two parameters

$$\bar{\alpha}, C$$

where C is the central charge.

Detailed calculations for the simplest model fixed by the equation $P^{d+2}_{\mu\nu} = 0$ are given in [6, 7], and in the general case of arbitrary s they are presented in ref. [12]. In particular, for the parameters $\bar{\alpha}$ and C it is obtained there that

$$\bar{\alpha} = -\frac{1}{2s}$$

$$C = 12\,\Gamma(s-1)\,\frac{\Gamma(d+1)}{\Gamma(d+s+1)}\left\{(-1)^s 1 - \frac{1}{4}(s+1)d(d+s-2) - \right.$$

$$\left. - \frac{1}{\Gamma(s+1)}(d-1)(d-2)\,\frac{\Gamma(d+s-1)}{\Gamma(d)}\left(\frac{d+s-1}{d+s-2} - \frac{1}{s+1}\right)\right\}$$

This coincides with the Kac formula [14] at $s = 2, 3$ and the corresponding models coincide with the minimal models of ref. [4]. Theories of a new class appear, however, at $s \geqslant 4$, for which the infinite-parameter symmetry is broken down to a six-parameter one.

The infinite-parameter symmetry models also can be obtained using this method. To this end it should be extended to the case of gauge theories and then the averages of nonlocal objects like the conformal string of paper [15] should be considered. This is possible for any D. Gauge-covariant fields will serve as analogs for the fields P_s. At $D = 2$, in particular, they are certain combinations of the fields P_s and of analogous fields generated by the operator products $T_{\mu\nu}(x_1)\,P_s(x_2)$. These combinations will obviously coincide with the zero fields that define the minimal models. This generalization of the method will be studied in the subsequent paper.

XI. WESS–ZUMINO MODEL

There are models where the fields P_s are generated by a conserved current rather that the energy-momentum tensor. The Thirring model is a simplest example [1]. Generally speaking, this is possible in the theories with no less than three different conformal-invariant functions of the $\langle j_\mu \ \varphi \ P_s \rangle$ type. The two-dimensional Wess-Zumino model

$$S = \frac{1}{4\lambda^2} \int tr\, (\partial_\mu g^{-1} \partial_\mu g)\, d^2 x +$$

$$+ \frac{k}{24\pi} \int \epsilon^{ABC}\, tr(g^{-1} \partial_A g g^{-1} \partial_B g g^{-1} \partial_C g)\, d^3 y$$

belongs to this class. There exists the conserved current

$$j_\mu = (\delta_{\mu\nu} + i\,\alpha \epsilon_{\mu\nu}) g^{-1}\, \partial_\nu g,$$

where

$$\alpha = \frac{1}{4\pi} \lambda^2 k \tag{11.2}$$

Let us consider equation

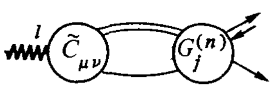
$$(\delta_{\mu\nu} + i\alpha \epsilon_{\mu\nu})\, \partial_\nu g(x) = g(x)\, j_\mu(x) \tag{11.3}$$

where the right-hand side is treated as the limit of operator product $g(x + \epsilon)$ $j_\mu(x)$ averaged over all directions of the vector ϵ_μ. It can be easily shown using the Ward identities that, along with $\partial_\mu g$, also the field P^{d+1}_μ of dimension $d + 1$ is involved in the expansion of this product in powers of the vector ϵ_μ. Availability of this field contradicts to (11.3).

It has to be demanded that

$$P^{d+1}_\mu = 0$$

The Green function of the field P^{d+1}_μ may be expressed in terms of the Green functions of the current, following the same method as the one used above for expressing the Green functions of the fields P_s in terms of the Green functions of the energy-momentum tensor. We have:

(11.4) $\qquad \langle P^{d+1}_\mu\, g \ldots g^{-1} \rangle = \underset{l=d+1}{res}$  $= 0$

where

$$\tilde{C}^a_{\mu,\nu}(x_1\, x_2\, x_3) = \langle P^l_\mu(x_1)\, \tilde{g}^{-1}(x_2)\, \tilde{j}_\nu(x_3) \rangle =$$

$$= (\delta_{\mu\nu} + i f \epsilon_{\mu\nu})(t^a)^{\alpha_3}_{\alpha_1} \delta^{\beta_1}_{\beta_3} \partial^{x_3}_{\nu}$$

$$\left[\lambda^{x_1}_{\tau}(x_2\, x_3)(x^2_{12})^{-\frac{l-d+1}{2}} (x^2_{13})^{-\frac{l+d-3}{2}} (x^2_{23})^{\frac{l+d-3}{2}}\right]$$

where f is an arbitrary constant, t^a is the algebra generator in the chosen representation of the chiral field $g(x) = [g(x)]^{\beta}_{\alpha}$.

Particularly, for the Green function $\langle j_\mu\, g\, g^{-1}\, j_\nu \rangle$ we have:

$$\langle P^{d+1}_{\mu}(x_1)\, g^{-1}(x_2)\, j^b_\nu(x_3)\rangle =$$

$$= \operatorname*{res}_{l=d+1} dy_1\, dy_2\, \tilde{C}^{\alpha}_{\mu,\nu_1}(x_1\, y_1\, y_2)\, \langle j^a_{\nu_1}(y_2)\, g(y_1)\, g^{-1}(x_2)\, j^b_\nu(x_3)\rangle =$$

The right-hand side integral is calculated by the above method (see [1, 3] for the Thrring model). The Ward identity needed for this is:

$$\partial^{x_3}_{\nu_1} \langle g(x_1)\, g^{-1}(x_2)\, j^a_{\nu_1}(x_3)\, j^b_\nu(x_4)\rangle =$$

$$= -\, \delta(x_{13})\, t^a \langle g(x_1)\, g^{-1}(x_2)\, j^b_\nu(x_4)\rangle$$

$$+\, \delta(x_{23})\, \langle g(x_1)\, g^{-1}(x_2)\, j^b_\nu(x_4)\rangle\, t^a\, +$$

$$+\, i\delta(x_{3y})\, f^{abc} \langle g(x_1)\, g^{-1}(x_2)\, j^c_\nu(x_4)\rangle$$

$$-\, \frac{c}{8\pi} (\delta_{\nu\rho} + i\,\beta\epsilon_{\nu\rho})\, \partial^{x_3}_{\rho}\, \delta(x_{34})\, \delta^{ab} \langle g(x_1)\, g^{-1}(x_2)\rangle$$

where β and c are certain constants (*),

$$\langle g(x_1)\, g^{-1}(x_2)\, j^a_\nu(x_3)\rangle =$$

$$= \frac{1}{2\pi} (\delta_{\nu\tau} + i\,\alpha\epsilon_{\nu\tau})\, \lambda^{x_3}_{\tau}(x_1\, x_2)\, \langle g(x_1)\, g^{-1}(x_2)\rangle.$$

From the calculation we obtain:

$$(\delta_{\lambda\tau} + i f \epsilon_{\lambda\tau}) \left\{ \frac{1}{2} C_g (\delta_{\nu\rho} + i\,\alpha\epsilon_{\nu\rho}) \frac{1}{x^2_{1y}} g_{\rho\tau}(x_{14}) + \right. \tag{11.5}$$

(*) The field P^{D-2} is an analog to the central charge C at $D > 2$, as before. Its contribution to the Ward identity has the form:

$$\partial^{x_3}_{\rho}\, \delta(x_{3y})\, \langle P^{D-2}(x_1)\, g(x_1)\, g^{-1}(x_2)\rangle$$

for $D = 2$ one has [16] that $C = k$.

(11.5)
$$-\frac{1}{2}\, d\, C_{\nu}(\delta_{\nu\rho} + i\,\alpha\,\epsilon_{\nu\rho})\,\lambda^{x_1}_{\rho}\,(x_2\, x_4)\lambda^{x}_{\tau 4}(x_1\, x_2) -$$
$$-\frac{1}{y}\, d\, C\,(\delta_{\nu\rho} + i\,\beta\,\epsilon_{\nu\rho})\,\frac{1}{x^2_{1y}}\, g_{\rho\tau}(x_{14})\Big\} = 0$$

where

$$C_g = t^a t^a, \quad f^{abc} f^{a'bc} = \delta^{aa'}\, C_{\nu}$$

Taking into account the relations

$$(\delta_{\lambda\tau} + i\,\epsilon_{\lambda\tau})\,(\delta_{\nu\rho} + i\,\epsilon_{\nu\rho})\,g_{\rho\tau}(x) = 2(\delta_{\lambda\tau} + i\,\epsilon_{\lambda\tau}\,)\; g_{\tau\nu}(x)$$
$$\frac{1}{y^2_{1y}}\;(\delta_{\nu\rho} + i\,\epsilon_{\nu\rho})\,g_{\rho\lambda}(y_{14}) =$$
$$= (\delta_{\nu\rho} + i\,\epsilon_{\nu\rho})\,(\delta_{\lambda\tau} + i\,\epsilon_{\lambda\tau})\,\lambda^{x_1}_{\rho}\,(x_4\, x_2)\,\lambda^{x}_{\tau 4}(x_1\, x_2)$$

one can determine: $\alpha = \beta = f = 1$. From this it follows that

(11.6)
$$d = \frac{2C_g}{C_{\nu} + C}$$

Note, that from the equality $\alpha = 1$ we have

$$\lambda^2 = \frac{4\pi}{k}$$

Therefore, the coupling constant in (11.1) is also calculated from the condition of conformal invariante. As it is known [16], the β-function of the model (11.1) has zero in this point.

From the equation

$$\langle P^{d+1}_{\mu}\; g^{-1} g g^{-1}\rangle =$$
$$= \int dy_1\, dy_2\; \tilde{C}^{a}_{\mu,\nu}(x_1\, y_1\, y_2)\,\langle j^a_{\nu}\,(y_2)\, g(y_1)\, g^{-1}(x_2) g(x_3) g^{-1}(x_4)\rangle =$$
$$= 0$$

and the Ward identity for the Green function $\langle j^a_{\mu}\; g g^{-1} g g^{-1}\rangle$ we have:

$$\Big\{\frac{1}{2}\, C_g\, \partial^{x_1}_{\tau} + d\Big[\frac{(x_{12})_{\tau}}{x^2_{12}}\;\; t^b_1\, t^b_2 - \frac{(x_{13})_{\tau}}{x^2_{13}}\, t^b_1\, t^b_3 +$$

$$+\frac{(x_{14})_\tau}{x_{14}^2}\, t_1^b\, t_4^b \quad \langle g(x_1)g^{-1}(x_2)g(x_3)g^{-1}(x_4)\rangle = 0$$

where t_i^b is the matrix acting on the index of the i-th field. This equation, as well as the result (11.6) is in agreement with the results in papers [5, 17].

The scale dimensions of the composite fields O_α are calculated from the equations $\langle P_\nu^{d+1}\; g^{-1}\; O_\alpha\rangle = 0$ and are in agreement with the known results, as well.

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